Darboux transformation and multi-soliton solutions of Two-Boson hierarchy

Ashok Das [a,b] and U. Saleem [a,c*]
[a] *Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627-0171, USA*
[b] *Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta 700064, India and*
[c] *Department of Physics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590,Pakistan*

We study Darboux transformations for the two boson (TB) hierarchy both in the scalar as well as in the matrix descriptions of the linear equation. While Darboux transformations have been extensively studied for integrable models based on $SL(2,R)$ within the AKNS framework, this model is based on $SL(2,R)\otimes U(1)$. The connection between the scalar and the matrix descriptions in this case implies that the generic Darboux matrix for the TB hierarchy has a different structure from that in the models based on $SL(2,R)$ studied thus far. The conventional Darboux transformation is shown to be quite restricted in this model. We construct a modified Darboux transformation which has a much richer structure and which also allows for multi-soliton solutions to be written in terms of Wronskians. Using the modified Darboux transformations, we explicitly construct one soliton/kink solutions for the model.

PACS numbers: 02.30.lk, 05.45.Yv, 11.10.-z

I. INTRODUCTION

During the past four decades there has been a lot of interest in the study of different classical and quantum integrable models from various points of view, for example, the Lax description, zero-curvature condition, the bi-Hamiltonian structures, multi-soliton solutions, conserved quantities etc. [1, 2]. Most of the familiar $1+1$ dimensional integrable systems can be described (in the matrix form) by the AKNS formalism, namely, a zero-curvature description for them is based on the symmetry group $SL(2,R)$. The multi-soliton solutions for such systems correspondingly have been constructed (among other methods) by the conventional Darboux transformations [3, 4] (in the scalar description) or in the matrix formalism (zero-curvature method) by Darboux transformations based on the symmetry group $SL(2,R)$[5].

The Two Boson (TB) hierarchy is an integrable system which in many ways is quite different from the other familiar integrable systems and has been studied extensively during the last two decades [6]-[11]. For example, the scalar Lax description of this model involves a nonstandard equation [6] and this system is known to be tri-Hamiltonian. In fact, the Hamiltonian structures of the bosonic as well as the supersymmetric TB hierarchy have been studied in [6, 7] and the bilinear and the trilinear forms of the hierarchy have been used in [8]-[10] to construct the multi-soliton solutions of this system. Furthermore, the zero-curvature formulation of this sytem (leading to the tri-Hamiltonian structure) does not fall within the conventional AKNS hierarchy based on $SL(2,R)$, rather it is based on the symmetry algebra $SL(2,R)\otimes U(1)$ [11]. As a result, one expects the Darboux transformation for this system to have new features and in this letter we show that this expectation is indeed true. In particular, we show that the conventional Darboux transformation in this case is quite restricted while a modified Darboux transformation leads to a much richer structure for solutions. Furthermore, the relation between the scalar and the matrix descriptions of the linear equations in this case leads to a Darboux matrix which is generically quite distinct from that studied so far within the context of the AKNS formalism. In section **II**, we recapitulate briefly various properties of interest of the TB hierarchy. In section **III**, we construct the conventional N-fold Darboux transformation (both for the scalar Lax equation and the zero-curvature condition) and express the multi-soliton solutions for this system in terms of Wronskians. In section **IV**, we introduce a modified N-fold Darboux transformation which has a much richer structure which also allows the multi-soliton solutions to be written in terms of the corresponding Wronskians. In the last section, we derive explicitly the one soliton/kink solutions of the TB hierarchy using the modified Darboux transformation.

---

* e-mail: das@pas.rochester.edu, usaleem@physics.pu.edu.pk, usaleem@pas.rochester.edu

II. RECAPITULATION OF THE TB HIERARCHY

The TB equation is a set of two $1+1$ dimensional equations given by [6, 7]

$$\begin{aligned} J_{0t} &= \left(2J_1 + J_0^2 - J_{0x}\right)_x, \\ J_{1t} &= \left(2J_0J_1 + J_{1x}\right)_x, \end{aligned} \tag{1}$$

where J_0, J_1 represent the dynamical variables (depending on t and x) of the system and the subscripts t and x represent partial derivatives with respect to these variables. The TB system (1) can be described as a nonstandard scalar Lax equation of the form

$$\frac{\partial L}{\partial t} = [L, M], \tag{2}$$

where the Lax operators L and M are given by

$$L = \partial - J_0 + \partial^{-1} J_1, \qquad M \equiv \left(L^2\right)_{\geq 1} = -\partial^2 + 2J_0\partial. \tag{3}$$

Here $\partial = \frac{\partial}{\partial x}$ and ∂^{-1} denotes the formal integration operation. The Lax operators L and M lead to the linear equations

$$L\psi = \lambda\psi, \qquad \psi_t = M\psi, \tag{4}$$

where λ is the constant spectral parameter and $\psi = \psi\,(x, t; \lambda)$ is a scalar wave function. Equation (4) generates the TB hierarchy of equations.

The linear equations (4) have the explicit forms

$$\psi_x = (J_0 + \lambda)\,\psi - \partial^{-1}(J_1\psi), \qquad \psi_t = (J_0 - \lambda)\,\psi_x + (J_1 - J_{0x})\,\psi, \tag{5}$$

where the second equation has been simplified using the first and the compatibility condition (i.e. $\psi_{xt} = \psi_{tx}$) of the linear system (5) is equivalent to the Lax equation (2) and leads to the TB system (1). For future use, we note that we can avoid the nonlocality in the first equation in (5) by operating with a derivative and, therefore, the pair of linear equations can equivalently be written as

$$\psi_{xx} = (J_0 + \lambda)\,\psi_x + (J_{0x} - J_1)\,\psi, \qquad \psi_t = (J_0 - \lambda)\,\psi_x + (J_1 - J_{0x})\,\psi. \tag{6}$$

It can be checked that the compatibility condition for (6) (i.e. $\psi_{xxt} = \psi_{txx}$) leads to the TB system of equations (1).

The TB hierarchy of equations (5) can also be expressed as a zero-curvature condition in terms of Lie algebra valued gauge fields A_0 and A_1 based on $SL(2,R)\otimes U(1)$ [11],

$$[\partial_x - A_1, \partial_t - A_0] \equiv \partial_t A_1 - \partial_x A_0 - [A_0, A_1] = 0, \tag{7}$$

where

$$A_1(x,t;\lambda) = \begin{pmatrix} J_0 + \lambda & -1 \\ J_1 & 0 \end{pmatrix}, \quad A_0(x,t;\lambda) = \begin{pmatrix} J_0^2 + J_1 - J_{0x} - \lambda^2 & \lambda - J_0 \\ J_0J_1 + J_{1x} - \lambda J_1 & J_1 \end{pmatrix}. \tag{8}$$

The zero-curvature condition (7) can be understood as the compatibility condition for the linear (matrix) equations

$$\partial_x\Psi(x,t;\lambda) = A_1(x,t;\lambda)\Psi(x,t;\lambda), \qquad \partial_t\Psi(x,t;\lambda) = A_0(x,t;\lambda)\Psi(x,t;\lambda), \tag{9}$$

where $\Psi(x,t;\lambda)$ denotes a $SL(2,R)\otimes U(1)$ Lie algebra valued matrix wave function. The connection between scalar linear equations (of the form (5)) and matrix equations (of the form (9)) can be traced to the work of Drinfeld and Sokolov [12, 13]. We note here for future use that the linear equations (6) can be related to the matrix equations (following [12, 13])

$$\partial_x\Psi(x,t;\lambda) = B_1(x,t;\lambda)\Psi(x,t;\lambda), \qquad \partial_t\Psi(x,t;\lambda) = B_0(x,t;\lambda)\Psi(x,t;\lambda), \tag{10}$$

where the $SL(2,R)\otimes U(1)$ valued gauge fields B_0, B_1 have the forms

$$B_1(x,t;\lambda) = \begin{pmatrix} J_0 + \lambda & J_{0x} - J_1 \\ 1 & 0 \end{pmatrix}, \quad B_0(x,t;\lambda) = \begin{pmatrix} J_0^2 + J_1 - \lambda^2 & (\lambda - J_0)\,(J_1 - J_{0x}) + (J_1 - J_{0x})_x \\ J_0 - \lambda & J_1 - J_{0x} \end{pmatrix}. \tag{11}$$

The compatibility condition of the matrix linear system (10) corresponds to the zero-curvature condition

$$[\partial_x - B_1, \partial_t - B_0] \equiv \partial_t B_1 - \partial_x B_0 - [B_0, B_1] = 0. \tag{12}$$

III. CONVENTIONAL DARBOUX TRANSFORMATION

In this section, we study systematically the properties of the conventional Darboux transformations for the scalar wave function ψ as well as the matrix wave function Ψ of the linear systems (6) and (9) (or equivalently (10)) associated with the TB hierarchy.

Let us recall that Darboux transformation allows one to construct a new solution of a Sturm-Liouville problem from a known solution. Alternatively, it can be used to obtain the solution of a Sturm-Liouville system from another (Sturm-Liouville) system. Conventionally, the Darboux transformation is defined as a transformation of the (scalar) wave function $\psi(x,t;\lambda)$ of the Sturm-Liouville system of the form

$$\psi(x,t;\lambda) \longmapsto \psi'(x,t;\lambda,\lambda_1) = \psi_x - \sigma_1\psi, \tag{13}$$

where

$$\sigma_1 = \frac{\psi_{1x}}{\psi_1}, \tag{14}$$

and ψ_1 is a solution of the original system for $\lambda = \lambda_1$. The new solution $\psi'(x,t;\lambda,\lambda_1)$ clearly satisfies the condition

$$\psi'(x,t;\lambda,\lambda_1)|_{\lambda=\lambda_1} = 0. \tag{15}$$

Requirement of covariance of the linear equation under the Darboux transformation (13) then determines the transformations of the potentials (other variables) in the problem. While it is not obvious in the form (13), Darboux transformation can be thought of as a very special gauge transformation which becomes manifest only in the context of the matrix description of a linear equation.

The conventional Darboux transformation (13) has been generalized to the AKNS family of integrable models (based on $SL(2,R)$) and is one of the well-known methods of obtaining multi-soliton solutions for such systems (see, for example, [3]-[5]). Let us now apply the conventional Darboux transformation (13) to the linear equations (6). Requirement of covariance of the linear equation (6) under (13) leads to the fact that $\psi'(x,t;\lambda,\lambda_1)$ satisfies the linear equations

$$\psi'_{xx} = (J'_0+\lambda)\,\psi'_x + (J'_{0x}-J'_1)\,\psi', \qquad \psi'_t = (J'_0-\lambda)\,\psi'_x + (J'_1-J'_{0x})\,\psi', \tag{16}$$

which in turn determines the transformations of the potentials (under the Darboux transformation) to be

$$J'_0 = J_0, \qquad J'_1 = J_1 - J_{0x} + 2\sigma_{1x} = J_1 - J_{0x} + 2\partial^2 \ln\psi_1. \tag{17}$$

It can be checked that the transformed variables in (17) do satisfy the TB equation (see (1))

$$J'_{0t} = \left(2J'_1 + (J'_0)^2 - J'_{0x}\right)_x, \qquad J'_{1t} = \left(2J'_0J'_1 + J'_{1x}\right)_x, \tag{18}$$

which also arises from the compatibility of (16). This shows that with some starting (seed) solution (J_0,J_1) of the two boson equation, one can construct new solutions (J'_0,J'_1) through the Darboux transformation. However, it is interesting to note from (17) that under the conventional Darboux transformation (13) the dynamical variable J_0 remains invariant so that the conventional Darboux transformation in this case is quite restrictive. Therefore, to summarize, we note that if the set (ψ,J_0,J_1) defines a solution of the linear equations (6), then the set $(\psi[1],J_0[1],J_1[1])$ defined by

$$\psi[1] \;=\; \psi' \equiv \psi_x - \frac{\psi_{1x}}{\psi_1}\psi = \frac{W(\psi_1,\psi)}{\psi_1}, \tag{19}$$

$$J_0[1] \;=\; J_0, \tag{20}$$

$$J_1[1] \;=\; J_1 - J_{0x} + 2\sigma_{1x} = J_1 - J_{0x} + 2\partial^2\ln\psi_1, \tag{21}$$

is the required one-fold Darboux transformation of the set under which (6) is covariant. Here

$$W(\psi_1,\psi) \equiv \begin{vmatrix} \psi_1 & \psi \\ \partial\psi_1 & \partial\psi \end{vmatrix} = \psi_1\psi_x - \psi_{1x}\psi, \tag{22}$$

is the usual Wronskian determinant of the two solutions.

The two-fold Darboux transformation is constructed in two steps. First, with a solution ψ_1 of the linear system (6) for the eigenvalue λ_1 we construct the set $(\psi[1], J_0[1], J_1[1])$ as discussed in (19)-(21). Then, with a solution $\psi_2[1]$ of (16) for eigenvalue λ_2, the two-fold Darboux transformation on the set (ψ, J_0, J_1) is determined to be

$$\begin{aligned} \psi[2] &\equiv (\psi[1])_x - \frac{(\psi_2[1])_x}{\psi_2[1]}\psi[1] = \frac{W(\psi_1,\psi_2,\psi)}{W(\psi_1,\psi_2)}, \\ J_0[2] &\equiv J_0[1] = J_0, \\ J_1[2] &\equiv J_1[1] - J_{0x}[1] + 2\partial^2 \ln \psi_2[1] = J_1 - 2J_0 + 2\partial^2 \ln W(\psi_1,\psi_2), \end{aligned} \tag{23}$$

where

$$W(\psi_1,\psi_2,\psi) \equiv \begin{vmatrix} \psi_1 & \psi_2 & \psi \\ \partial\psi_1 & \partial\psi_2 & \partial\psi \\ \partial^2\psi_1 & \partial^2\psi_2 & \partial^2\psi \end{vmatrix}, \tag{24}$$

and $W(\psi_1,\psi_2)$ is defined in (22). This procedure can be iterated N times leading to the N-fold Darboux transformation on the set (ψ, J_0, J_1) in the form

$$\begin{aligned} \psi[N] &= \frac{W(\psi_1,\psi_2,\ldots,\psi_N,\psi)}{W(\psi_1,\psi_2,\ldots,\psi_N)}, \\ J_0[N] &= J_0, \\ J_1[N] &= J_1 - NJ_{0x} + 2\partial^2 \ln W(\psi_1,\psi_2,\ldots,\psi_N), \end{aligned} \tag{25}$$

where

$$W(\psi_1,\psi_2,\ldots,\psi_N,\psi) = \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_N & \psi \\ \partial\psi_1 & \partial\psi_2 & \cdots & \partial\psi_N & \partial\psi \\ \partial^2\psi_1 & \partial^2\psi_2 & \cdots & \partial^2\psi_N & \partial^2\psi \\ \vdots & \vdots & \ddots \vdots & \vdots & \vdots \\ \partial^N\psi_1 & \partial^N\psi_2 & \cdots & \partial^N\psi_N & \partial^N\psi \end{vmatrix}. \tag{26}$$

It is worth noting here that under a conventional Darboux transformation (13) one of the dynamical variables, J_1, transforms while the second dynamical variable, J_0, remains invariant. If we set $J_0 = 0$ and $J_1 = u$, the TB hierarchy (6) is known to reduce to the KdV hierarchy and in this case the N-fold Darboux transformation (25) of the dynamical variable J_1 indeed reduces to the N-fold Darboux transformation of the KdV variable u.

The Darboux transformations for the matrix solution Ψ of the linear system (9) or (10) are given in terms of a 2×2 matrix $D(x,t;\lambda,\lambda_1)$, called the Darboux matrix. For a general discussion on Darboux matrix approach see e.g. [5]. However, since we have already determined the (conventional) Darboux transformation for the scalar linear equation (6) (see (19)-(21)), the Darboux matrix can be simply constructed following the connection between the scalar and the matrix descriptions for this system [13] and this brings out some new features. For example, if we consider the linear matrix equation (9), the one-fold Darboux transformation of the matrix wave function has the form

$$\Psi'(x,t;\lambda,\lambda_1) = D(x,t;\lambda,\lambda_1)\Psi(x,t;\lambda), \tag{27}$$

and the Darboux matrix has the form

$$D(x,t;\lambda,\lambda_1) = \begin{pmatrix} J_0 + \lambda - \sigma_1 & -1 \\ -J_{0x} + J_1 + \sigma_{1x} & -\sigma_1 \end{pmatrix}, \tag{28}$$

where σ_1 is defined in (14). Furthermore, using (8) and (9), we note that the Darboux transformation (27) for the wave function can be equivalently written as (which is simpler for calculations)

$$\Psi'(x,t;\lambda,\lambda_1) \equiv D(x,t;\lambda,\lambda_1)\Psi(x,t;\lambda) = (\partial - M(x,t;\lambda_1))\,\Psi(x,t;\lambda), \tag{29}$$

where the matrix $M(x,t;\lambda_1)$ is given by (here one uses the equation satisfied by σ_1 to simplify the matrix into the following form)

$$M(x,t;\lambda_1) = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_1^2 - (J_0+\lambda_1)\sigma_1 + J_1 & \sigma_1 \end{pmatrix}. \tag{30}$$

It is worth pointing out here that the Darboux matrix in the AKNS framework (based on $SL(2,R)$) has the generic form

$$D = \lambda \mathbb{1} - M, \tag{31}$$

where $\mathbb{1}$ denotes the 2×2 identity matrix. However, it can be checked that a Darboux matrix such as in (31) does not work in the case of the TB hierarchy (and, in fact, can be simply understood from the connection between the scalar and the matrix descriptions). On the other hand, the Darboux matrix for the TB hierarchy has the generic form as given in (29).

The requirement of covariance of (9), requires that the transformed matrix wave function $\Psi'(x,t;\lambda,\lambda_1)$ satisfies

$$\frac{\partial \Psi'}{\partial x} = A_1' \Psi', \qquad \frac{\partial \Psi'}{\partial t} = A_0' \Psi', \tag{32}$$

where the transformation of the gauge potentials (fields) A_1' and A_0' are given by

$$A_1' = D A_1 D^{-1} + D_x D^{-1}, \qquad A_0' = D A_0 D^{-1} + D_t D^{-1}. \tag{33}$$

The zero-curvature condition (7) is covariant under a gauge transformation so that the transformed potentials (A_1', A_0') lead to a vanishing field strength (curvature). Under the Darboux transformation (29) and (33), the set (Ψ, A_1, A_0) maps into (Ψ', A_1', A_0').

Similarly, the matrix Darboux transformation for the linear matrix equation (10) can be written in the generic form (29) with

$$M(x,t;\lambda_1) = \begin{pmatrix} \sigma_1 & -\sigma_1^2 + (J_0+\lambda_1)\sigma_1 + (J_{0x} - J_1) \\ 0 & \sigma_1 \end{pmatrix}. \tag{34}$$

with the gauge fields (B_1, B_0) transforming as

$$B_1' = D B_1 D^{-1} + D_x D^{-1}, \qquad B_0' = D B_0 D^{-1} + D_t D^{-1}. \tag{35}$$

In either case, it is easy to determine that (33) or (35) lead to

$$J_0' = J_0, \qquad J_1' = J_1 - J_{0x} + 2\sigma_{1x} = J_1 - J_{0x} + 2\partial^2 \ln \psi_1, \tag{36}$$

which coincides with (17). The matrix Darboux transformation can also be iterated N times to obtain expressions for the N-fold Darboux transformation on the sets (Ψ, A_1, A_0) and (Ψ, B_1, B_0) which lead to the same results as in the scalar case.

IV. MODIFIED DARBOUX TRANSFORMATION

As we have seen in the last section, the conventional Darboux transformation, in the case of the TB hierarchy, generates new solutions starting with known ones (see (19)-(21)). However, it seems to be quite restricted in the sense that the variable J_0 does not seem to transform at all under such a transformation. In this section we present a modified Darboux transformation of the linear system (6) such that the requirement of covariance of the linear system (6) allows one to create new solutions of the TB system (1) which are more general (namely, it allows for J_0 to transform).

Let us consider the transformation of the linear system (6) defined by

$$\psi(x,t;\lambda) \longmapsto \psi'(x,t;\lambda,\lambda_1) = \psi - \sigma_1^{-1}\psi_x. \tag{37}$$

The new solution $\psi'(x,t;\lambda,\lambda_1)$ vanishes at $\lambda=\lambda_1$, namely, $\psi'(x,t;\lambda,\lambda_1)|_{\lambda=\lambda_1} = 0$ as in the conventional Darboux transformation in (15). In fact, we note that the modified Darboux transformation (37) is related to the conventional Darboux transformation (13) through a (space-time dependent) factor $(-\sigma_1^{-1})$, namely,

$$\psi'_{\text{modified}} = -\sigma_1^{-1}\psi'_{\text{conventional}}. \tag{38}$$

However, this is sufficient to modify the character of the transformations for the potentials.

Requiring the transformed function $\psi'(x,t;\lambda,\lambda_1)$ to satisfy the linear system (covariance)

$$\psi'_{xx} = (J_0' + \lambda)\,\psi'_x + (J_{0x}' - J_1')\,\psi', \qquad \psi'_t = (J_0' - \lambda)\,\psi'_x + (J_1' - J_{0x}')\,\psi', \tag{39}$$

determines the transformations of J_0, J_1 under the modified Darboux transformation to be

$$
\begin{aligned}
J_0' &= J_0 - \frac{\sigma_{1x}}{\sigma_1}, && (40)\\
J_1' &= J_1 - J_{0x} + \sigma_{1x}. && (41)
\end{aligned}
$$

It is interesting to note that, unlike the case of the conventional Darboux transformation (see (19)-(21)), here both J_0 and J_1 transform under the action of the modified Darboux transformation (37) and, therefore, it has a richer structure. These new potentials satisfy the evolution equation (which follows from the compatibility of (39))

$$
J_{0t}' = \left(2J_1' + (J_0')^2 - J_{0x}'\right)_x, \qquad J_{1t}' = (2J_0'J_1' + J_{1x}')_x, \tag{42}
$$

which is the TB equation (1) in the new variables. Namely, (J_0', J_1') can be thought of as new solutions of the TB equation starting from known ones (J_0, J_1).

In other words we can say that if the set (ψ, J_0, J_1) represents a solution of the linear system (6), then the set $(\psi[1], J_0[1], J_1[1])$ given by

$$
\begin{aligned}
\psi[1] &\equiv \psi - \frac{\psi_1}{\psi_{1x}}\psi_x = \frac{W(\psi, \psi_1)}{\psi_{1x}}, && (43)\\
J_0[1] &\equiv J_0 - \frac{\sigma_{1x}}{\sigma_1} = J_0 - \partial \ln\left(\frac{\psi_{1x}}{\psi_1}\right), && (44)\\
J_1[1] &\equiv J_1 - J_{0x} + \sigma_{1x} = J_1 - J_{0x} + \partial^2 \ln \psi_1, && (45)
\end{aligned}
$$

the one-fold (modified) Darboux transformation under which (6) is covariant. Following the discussion of the last section, the two-fold Darboux transformation of the set (ψ, J_0, J_1) can be constructed and is given by

$$
\begin{aligned}
\psi[2] &\equiv \psi[1] - \frac{\psi_2[1]}{(\psi_2[1])_x}(\psi[1])_x = \frac{W(\psi_1, \psi_2, \psi)}{W(\psi_{1x}, \psi_{2x})}, && (46)\\
J_0[2] &\equiv J_0[1] - \partial \ln\left(\frac{(\psi_1[1])_x}{\psi_1[1]}\right) = J_0 - \partial \ln\left(\frac{W(\psi_{1x}, \psi_{2x})}{W(\psi_1, \psi_2)}\right), && (47)\\
J_1[2] &\equiv J_1[1] - J_{0x}[1] + \partial^2 \ln \psi_2[1] = J_1 - 2J_{0x} + \partial^2 \ln W(\psi_1, \psi_2), && (48)
\end{aligned}
$$

where ψ_i with $i = 1, 2$ denote solutions of the linear system (6) corresponding to the eigenvalues λ_i and the Wronskians are defined in (22) and (24).

The (modified) Darboux transformation can be iterated N-times to obtain the N-fold Darboux transformation of the set (ψ, J_0, J_1) which has the form

$$
\begin{aligned}
\psi[N] &= \frac{W(\psi_1, \psi_2, \ldots, \psi_N, \psi)}{W(\psi_{1x}, \psi_{2x}, \ldots, \psi_{Nx})}, && (49)\\
J_0[N] &= J_0 - \partial \ln\left(\frac{W(\psi_{1x}, \psi_{2x}, \ldots, \psi_{Nx})}{W(\psi_1, \psi_2, \ldots, \psi_N)}\right), && (50)\\
J_1[N] &= J_1 - NJ_{0x} + \partial^2 \ln W(\psi_1, \psi_2, \ldots, \psi_N), && (51)
\end{aligned}
$$

with the Wronskian defined in (26). From (25) as well as (51) it follows that in either case the multi-soliton solutions can be expressed in terms of Wronskians.

Following the connection between the scalar and matrix descriptions for the TB hierarchy [13], we can now define the modified Darboux transformation for the matrix wavefunction Ψ of the linear system (9) or (10). In fact, the generic form of the transformation in this case can be written as (compare with (29) and (38))

$$
\Psi'(x, t; \lambda, \lambda_1) \equiv D(x, t; \lambda, \lambda_1)\Psi(x, t; \lambda) = \left(\mathbb{1} - M^{-1}\partial\right)\Psi(x, t; \lambda), \tag{52}
$$

where M has the form given in (30) or (34) depending on the matrix description. The transformation of the (gauge) potentials continues to be given by (33) or (35) for the covariance of the linear matrix equation and this leads explicitly to (40) and (41).

V. EXPLICIT ONE SOLITON/KINK SOLUTIONS

We can now calculate explicitly the one soliton/kink solutions of the TB system (1), using the Darboux transformations. For example, we note that with the seed (trivial) solutions $J_0 = 0$ and $J_1 = 0$, the linear system has the general solution of the form

$$\psi(x,t;\lambda) \;=\; a + b e^{\lambda(x-\lambda t)}, \tag{53}$$

where a, b are constants and λ is the spectral parameter. It follows now that

$$\sigma_1 = \frac{\psi_{1x}}{\psi_1} = \frac{\lambda_1 b e^{\lambda_1(x-\lambda_1 t)}}{a + b e^{\lambda_1(x-\lambda_1 t)}}, \tag{54}$$

and the modified Darboux transformations (44) and (45) lead to

$$J_0[1] \;=\; -\frac{\lambda_1 a}{a + b e^{\lambda_1(x-\lambda_1 t)}}, \tag{55}$$

$$J_1[1] \;=\; \frac{\lambda_1^2 b}{\left(a e^{-\left(\frac{\lambda_1 x - \lambda_1^2 t}{2}\right)} + b e^{\left(\frac{\lambda_1 x - \lambda_1^2 t}{2}\right)}\right)^2}. \tag{56}$$

The solutions of the TB equation in (55) and (56) can be plotted, say for example for $\lambda_1 = 1, a = 1, b = 2$, and are shown in Fig. 1 and Fig. 2 respectively. We see that the solution (55) for J_0 represents a one kink solution (Fig. 1) while the solution (56) for J_1 corresponds to a one soliton solution (Fig. 2).

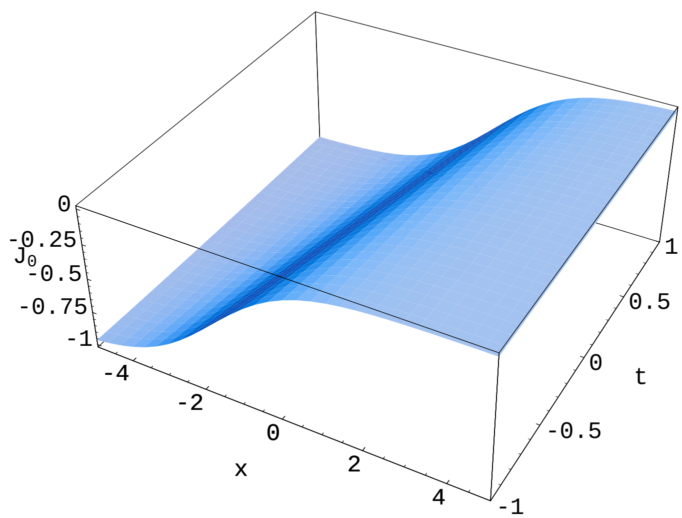

FIG. 1: One kink solution $J_0[1]$ of the TB system given by equation (55) with $\lambda_1 = 1, a = 1, b = 2$.

This soliton/kink behavior of the solutions is manifest analytically for $a = 1$ and $b = 1$ in which case (55) and (56) take the forms

$$J_0[1] \;=\; -\frac{\lambda_1}{2} + \frac{\lambda_1}{2}\tanh\left(\frac{\lambda_1 x - \lambda_1^2 t}{2}\right), \tag{57}$$

$$J_1[1] \;=\; \frac{\lambda_1}{4}\mathrm{sech}^2\left(\frac{\lambda_1 x - \lambda_1^2 t}{2}\right). \tag{58}$$

These particular solutions (57) and (58) coincide with the solutions of the TB system obtained in [8]-[10] through Hirota's method. We note here that we can also construct soliton/kink solutions of the TB equation using (20) and (21). However, in this case, these solutions would only be for the variable J_1.

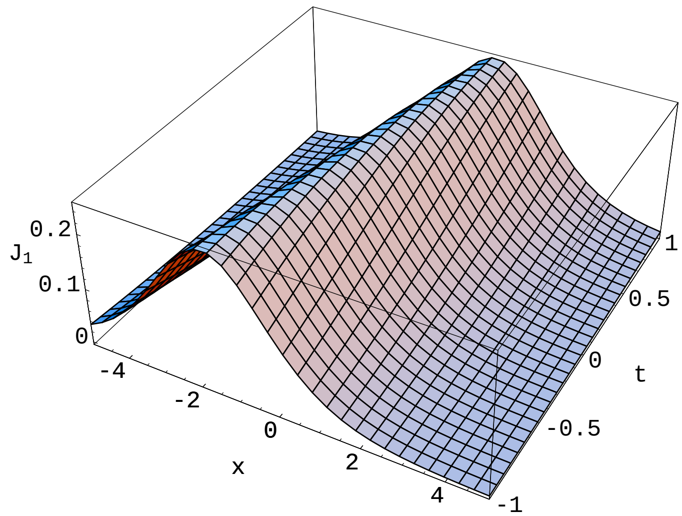

FIG. 2: One soliton solution $J_1[1]$ of the TB system given by equation (56) with $\lambda_1 = 1, a = 1, b = 2$.

VI. CONCLUDING REMARKS

In this paper we have studied the Darboux transformations for the TB hierarchy both in the scalar as well as the matrix descriptions of the linear equation. While the Darboux transformations have been extensively studied within the context of AKNS systems based on $SL(2, R)$, this is the first model where the symmetry group corresponds to $SL(2, R) \otimes U(1)$. The relation between the scalar and the matrix descriptions in the present case implies that the generic form of the Darboux transformation in the matrix case is different for the TB hierarchy. We show that the conventional Darboux transformation is quite restricted in this case in the sense that one of the dynamical variables remains inert under the transformation. We construct a modified Darboux transformation which has a richer structure and allows for change in both the dynamical variables of the theory. We show that in both the conventional as well as the modified Darboux transformations, the N-fold transformations (multi-soliton solutions) can be expressed in terms of Wronskians. We have constructed explicit one soliton/kink solutions for this model using the modified Darboux transformation. The generalization of these results to the case of supersymmetric TB hierarchy is currently under study.

Acknowledgement:
US would like to thank the theory group at the Department of Physics and Astronomy, University of Rochester for hospitality and the Higher Education Commission, Pakistan for a research fellowship which made this work possible. He would also like to thank Mahmood ul Hassan for discussions and valuable comments. This work was supported in part by US DOE Grant number DE-FG 02-91ER40685.

---

[1] M. J. Ablowitz and P. A. Clarkson, *"Solitons, nonlinear evolution equations and inverse scattering"*, Lond. Math. Soc. Lect. Note Ser. **149**, 1 (1991)
L. D. Faddeev and L. A. Takhtajan, *"Hamiltonian methods in the theory of solitons"*, Springer, Berlin Germany (1987)
L. A. Dickey, *"Soliton equations and Hamiltonian systems"*, World Scientific, Singapore (2003) 408 P
R. K. Dodd, H. C. Morris, J. C. Eilbeck and J. D. Gibbon, *"Solitons and nonlinear wave equations"*, London, UK: Academic (1982) 630 P

[2] O. Babelon, D. Bernard and M. Talon, *"Introduction to classical integrable systems"*, Cambridge, UK: Univ. Pr. (2003) 602 P
G. L. Lamb, *"Elements of soliton theory"*, New York, USA: Wiley (1980) 289 P. (Pure and Applied Mathematics)
A. Das, *"Integrable models"*, World Scientific Singapore (1989) 339 P. (World Scientific Lecture Notes in Physics, Volume 39)

[3] G. Darboux, *"Sur une proposition relative aux équations linéaires (On a proposition relative to linear equations)"*, C. R. Acad. Sci. Paris **94** (1882) 1456-1459.
G. Darboux, *"Lecons sur la théorie générale des surfaces Volume 2"*, Gauthier-Villars, Paris (1915) 214-215.

[4] V. B. Matveev and M. A. Salle, *"Darboux transformations and soliton"*, Springer-Verlag (1991).
[5] C. H. Gu, H. Hu and Z. Zhou, *"Darboux transformations in integrable systems, theory and their applications to geometry"*, Berlin: Springer (2005).
[6] B. A. Kupershmidt, *"Mathematics of dispersive water waves"*, Commun. Mat. Phys. **99** (1985) 51-73.
[7] J. C. Brunelli and A. Das, *"The supersymmetric two boson hierarchies"*, Phys. Lett. B **337** (1994) 303-307.
J. C. Brunelli and A. Das, *"Properties of nonlocal charges in the supersymmetric two boson hierarchy"*, Phys. Lett. B **354** (1994) 307-314.
J. C. Brunelli and A. Das, *"Supersymmetric two boson equation, its reductions and the nonstandard supersymmetric KP hierarchy"*, Int. J. Mod. Phys. A **10** (1995) 4563-4599.
[8] J. Satsuma, K. Kajiwara, J. Matsukidaira and J. Hietarinta, *"Solutions of the Broer-Kaup system through its trilinear form"*, J. Phys. Soc. Jpn. **61** (1992) 3096-3102.
[9] J. Matsukidaira and J. Satsuma, *"Soliton equations expressed by trilinear forms and their solutions "*, Phys. Lett. A **147** (1990) 467-471.
[10] P.G. Estevéz, P.R. Gordoa, L. M. Alonso and E. M. Reus, *"On the Characterization of a new soliton sector in the classical Boussinesq system"*, Inverse Problems **10** (1994) L23-L27.
[11] A. Das and S. Roy, *"The zero-curvature formulation of TB, sTB hierarchy and topological algebras"*, Mod. Phys. Lett. **A 11** (1996) 1317-1330.
[12] V. G. Drinfeld and V. V. Sokolov,*"Lie algebras and equations of Korteweg-de Vries type"*, J. Sov. Math. **30** (1985) 1975-2035; Sov. Math. Dokl. **23** (1981) 457-462.
[13] H. Aratyn, A. Das, C. Rasinariu and A. H. Zimerman, *"Zero-curvature formalism in superspace"*, Lect. Notes in Phys. **502** (1998) 212-240.